# Chapter 3

## Arithmetic for Computers

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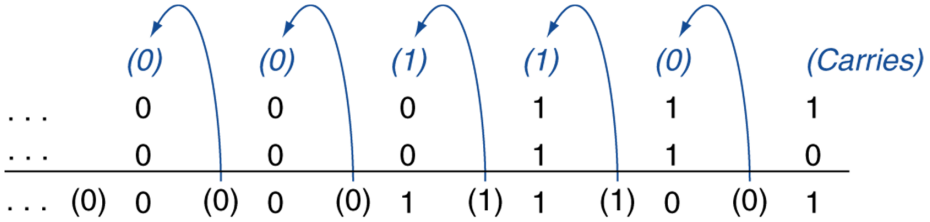
§3.1 Introduction

- Operations on integers
  - Addition and subtraction
  - Multiplication and division
  - Dealing with overflow
- Floating-point real numbers
  - Representation and operations

# Integer Addition

- Example: 7 + 6




- Overflow if result out of range
  - Adding +ve and –ve operands, no overflow
  - Adding two +ve operands
    - Overflow if result sign is 1
  - Adding two –ve operands
    - Overflow if result sign is 0



# Integer Subtraction

- Add negation of second operand
- Example: 7 – 6 = 7 + (–6)

| | |
|---|---|
| +7: | 0000 0000 … 0000 0111 |
| –6: | 1111 1111 … 1111 1010 |
| +1: | 0000 0000 … 0000 0001 |

- Overflow if result out of range
  - Subtracting two +ve or two –ve operands, no overflow
  - Subtracting +ve from –ve operand
    - Overflow if result sign is 0
  - Subtracting –ve from +ve operand
    - Overflow if result sign is 1

# Dealing with Overflow

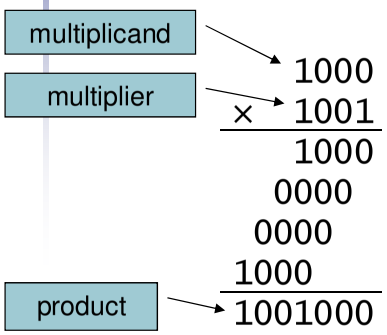
- Some languages (e.g., C) ignore overflow
  - Use MIPS `addu`, `addui`, `subu` instructions
- Other languages (e.g., Ada, Fortran) require raising an exception
  - Use MIPS `add`, `addi`, `sub` instructions
  - On overflow, invoke exception handler
    - Save PC in exception program counter (EPC) register
    - Jump to predefined handler address
    - `mfc0` (move from coprocessor reg) instruction can retrieve EPC value, to return after corrective action



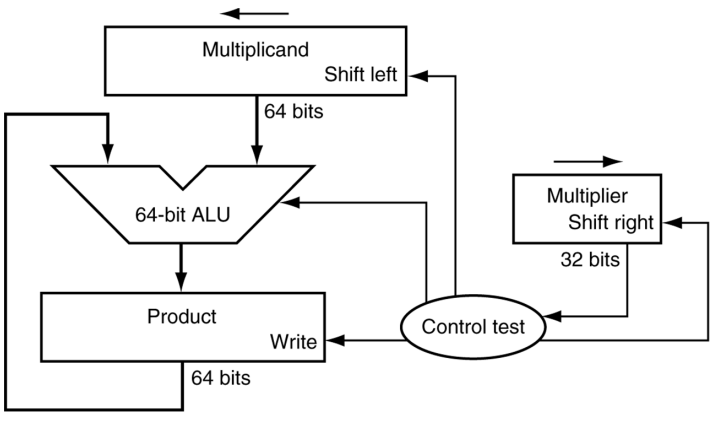


# Multiplication

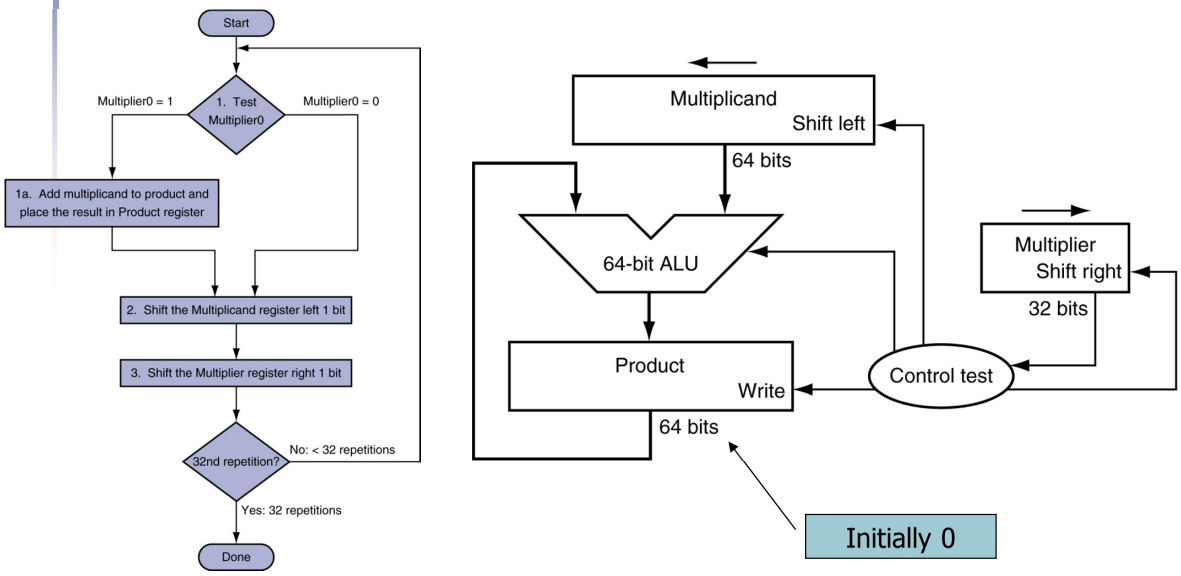
- Start with long-multiplication approach




Length of product is the sum of operand lengths




# Multiplication Hardware






# MIPS Multiplication

- Two 32-bit registers for product
  - HI: most-significant 32 bits
  - LO: least-significant 32-bits
- Instructions
  - `mult rs, rt / multu rs, rt`
    - 64-bit product in HI/LO
  - `mfhi rd / mflo rd`
    - Move from HI/LO to rd
    - Can test HI value to see if product overflows 32 bits
  - `mul rd, rs, rt`
    - Least-significant 32 bits of product –> rd

# MIPS Division

- Use HI/LO registers for result
  - HI: 32-bit remainder
  - LO: 32-bit quotient
- Instructions
  - `div rs, rt` / `divu rs, rt`
  - No overflow or divide-by-0 checking
    - Software must perform checks if required
  - Use `mfhi`, `mflo` to access result



# Floating Point



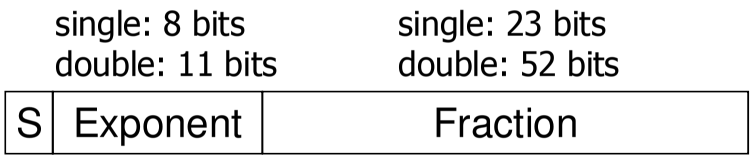
- Representation for non-integral numbers
  - Including very small and very large numbers
- Like scientific notation
  - $-2.34 \times 10^{56}$ ← normalized
  - $+0.002 \times 10^{-4}$ ← not normalized
  - $+987.02 \times 10^{9}$ ← not normalized
- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- Types `float` and `double` in C

# Floating Point Standard

- Defined by IEEE Std 754-1985
- Developed in response to divergence of representations
  - Portability issues for scientific code
- Now almost universally adopted
- Two representations
  - Single precision (32-bit)
  - Double precision (64-bit)



# IEEE Floating-Point Format




$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 ⇒ non-negative, 1 ⇒ negative)
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary-point 1 bit, so no need to represent it explicitly (hidden bit)
  - Significand is Fraction with the "1." restored
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned
  - Single: Bias = 127; Double: Bias = 1203

# Single-Precision Range

- Exponents 00000000 and 11111111 reserved
- Smallest value
  - Exponent: 00000001
    ⇒ actual exponent = 1 – 127 = –126
  - Fraction: 000…00 ⇒ significand = 1.0
  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
  - exponent: 11111110
    ⇒ actual exponent = 254 – 127 = +127
  - Fraction: 111…11 ⇒ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$



# Double-Precision Range

- Exponents 0000…00 and 1111…11 reserved
- Smallest value
  - Exponent: 00000000001
    ⇒ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 ⇒ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
  - Exponent: 11111111110
    ⇒ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 ⇒ significand ≈ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

# Floating-Point Precision

- Relative precision
  - all fraction bits are significant
  - Single: approx $2^{-23}$
    - Equivalent to $23 \times \log_{10}2 \approx 23 \times 0.3 \approx 6$ decimal digits of precision
  - Double: approx $2^{-52}$
    - Equivalent to $52 \times \log_{10}2 \approx 52 \times 0.3 \approx 16$ decimal digits of precision



# Floating-Point Example

- Represent –0.75
  - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
  - S = 1
  - Fraction = $1000\ldots00_2$
  - Exponent = –1 + Bias
    - Single: –1 + 127 = 126 = $01111110_2$
    - Double: –1 + 1023 = 1022 = $01111111110_2$
- Single: 10111111101000…00
- Double: 10111111111101000…00

# Floating-Point Example

- What number is represented by the single-precision float

  1100000010100000…00

  - S = 1
  - Fraction = $01000\ldots00_2$
  - Fxponent = $10000001_2 = 129$
- $x = (-1)^1 \times (1 + 01_2) \times 2^{(129 - 127)}$

  $= (-1) \times 1.25 \times 2^2$

  $= -5.0$



# Floating-Point Addition

- Consider a 4-digit decimal example
  - $9.999 \times 10^1 + 1.610 \times 10^{-1}$
- 1. Align decimal points
  - Shift number with smaller exponent
  - $9.999 \times 10^1 + 0.016 \times 10^1$
- 2. Add significands
  - $9.999 \times 10^1 + 0.016 \times 10^1 = 10.015 \times 10^1$
- 3. Normalize result & check for over/underflow
  - $1.0015 \times 10^2$
- 4. Round and renormalize if necessary
  - $1.002 \times 10^2$

# Floating-Point Addition

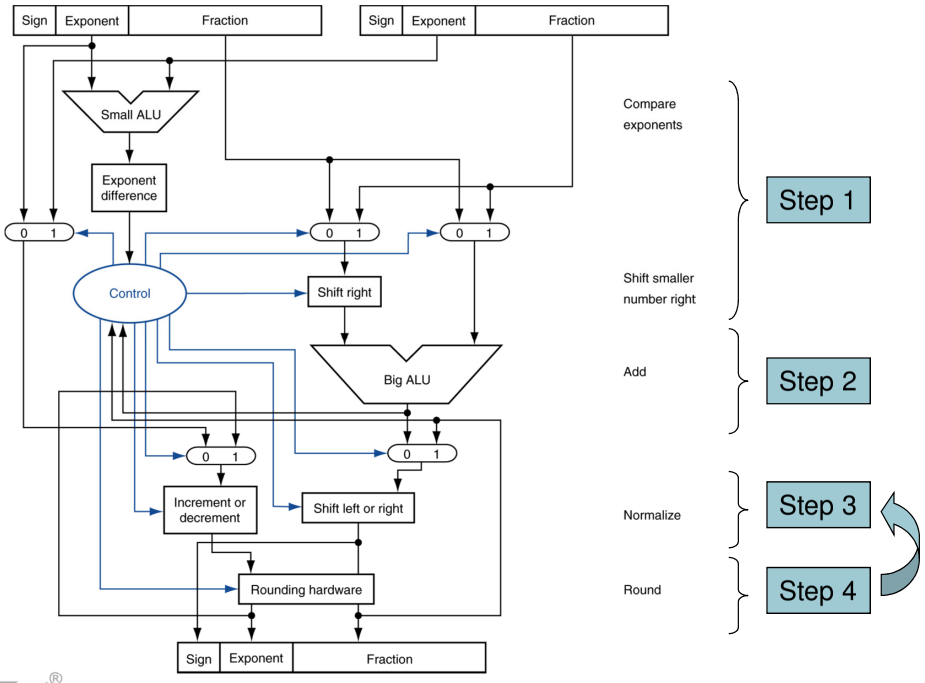
- Now consider a 4-digit binary example
  - $1.000_2 \times 2^{-1} + -1.110_2 \times 2^{-2}$ (0.5 + –0.4375)
- 1. Align binary points
  - Shift number with smaller exponent
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1}$
- 2. Add significands
  - $1.000_2 \times 2^{-1} + -0.111_2 \times 2^{-1} = 0.001_2 \times 2^{-1}$
- 3. Normalize result & check for over/underflow
  - $1.000_2 \times 2^{-4}$, with no over/underflow
- 4. Round and renormalize if necessary
  - $1.000_2 \times 2^{-4}$ (no change) = 0.0625



# FP Adder Hardware

- Much more complex than integer adder
- Doing it in one clock cycle would take too long
  - Much longer than integer operations
  - Slower clock would penalize all instructions
- FP adder usually takes several cycles
  - Can be pipelined

# FP Adder Hardware






# FP Arithmetic Hardware

- FP multiplier is of similar complexity to FP adder
  - But uses a multiplier for significands instead of an adder
- FP arithmetic hardware usually does
  - Addition, subtraction, multiplication, division, reciprocal, square-root
  - FP ↔ integer conversion
- Operations usually takes several cycles
  - Can be pipelined

# FP Instructions in MIPS

- FP hardware is coprocessor 1
  - Adjunct processor that extends the ISA
- Separate FP registers
  - 32 single-precision: $f0, $f1, … $f31
  - Paired for double-precision: $f0/$f1, $f2/$f3, …
    - Release 2 of MIPs ISA supports 32 × 64-bit FP reg's
- FP instructions operate only on FP registers
  - Programs generally don't do integer ops on FP data, or vice versa
  - More registers with minimal code-size impact
- FP load and store instructions
  - `lwc1`, `ldc1`, `swc1`, `sdc1`
    - e.g., `ldc1 $f8, 32($sp)`



# FP Instructions in MIPS

- Single-precision arithmetic
  - `add.s`, `sub.s`, `mul.s`, div.s
    - e.g., `add.s $f0, $f1, $f6`
- Double-precision arithmetic
  - `add.d`, `sub.d`, `mul.d`, `div.d`
    - e.g., `mul.d $f4, $f4, $f6`
- Single- and double-precision comparison
  - `c.`*xx*`.s`, `c.`*xx*`.d` (*xx* is `eq`, `lt`, `le`, …)
  - Sets or clears FP condition-code bit
    - e.g. `c.lt.s $f3, $f4`
- Branch on FP condition code true or false
  - `bc1t`, `bc1f`
    - e.g., `bc1t TargetLabel`

# FP Example: °F to °C

- C code:

```
float f2c (float fahr) {
  return ((5.0/9.0)*(fahr - 32.0));
}
```

  - fahr in $f12, result in $f0, literals in global memory space

- Compiled MIPS code:

```
f2c: lwc1  $f16, const5($gp)
     lwc2  $f18, const9($gp)
     div.s $f16, $f16, $f18
     lwc1  $f18, const32($gp)
     sub.s $f18, $f12, $f18
     mul.s $f0,  $f16, $f18
     jr    $ra
```



# Interpretation of Data

**The BIG Picture**

- Bits have no inherent meaning
  - Interpretation depends on the instructions applied
- Computer representations of numbers
  - Finite range and precision
  - Need to account for this in programs

# Associativity

- Parallel programs may interleave operations in unexpected orders
  - Assumptions of associativity may fail

| | | (x+y)+z | x+(y+z) |
|---|---|---|---|
| x | -1.50E+38 | | -1.50E+38 |
| y | 1.50E+38 | 0.00E+00 | |
| z | 1.0 | 1.0 | 1.50E+38 |
| | | 1.00E+00 | 0.00E+00 |

- Need to validate parallel programs under varying degrees of parallelism





# x86 FP Architecture

- Originally based on 8087 FP coprocessor
  - 8 × 80-bit extended-precision registers
  - Used as a push-down stack
  - Registers indexed from TOS: ST(0), ST(1), …
- FP values are 32-bit or 64 in memory
  - Converted on load/store of memory operand
  - Integer operands can also be converted on load/store
- Very difficult to generate and optimize code
  - Result: poor FP performance

# x86 FP Instructions

| Data transfer | Arithmetic | Compare | Transcendental |
|---|---|---|---|
| FILD mem/ST(i)<br>FISTP mem/ST(i)<br>FLDPI<br>FLD1<br>FLDZ | FIADDP mem/ST(i)<br>FISUBRP mem/ST(i)<br>FIMULP mem/ST(i)<br>FIDIVRP mem/ST(i)<br>FSQRT<br>FABS<br>FRNDINT | FICOMP<br>FIUCOMP<br>FSTSW AX/mem | FPATAN<br>F2XMI<br>FCOS<br>FPTAN<br>FPREM<br>FPSIN<br>FYL2X |

- Optional variations
  - I: integer operand
  - P: pop operand from stack
  - R: reverse operand order
  - But not all combinations allowed



# Concluding Remarks



- ISAs support arithmetic
  - Signed and unsigned integers
  - Floating-point approximation to reals
- Bounded range and precision
  - Operations can overflow and underflow
- MIPS ISA
  - Core instructions: 54 most frequently used
    - 100% of SPECINT, 97% of SPECFP
  - Other instructions: less frequent